逆強単調写像に関する変分不等式問題を扱った Badriev と Zadvornov の結果の一考察

Note on Badriev and Zadvornov’s results for variational inequality problems for inverse-strongly monotone mappings

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Abstract

In [3], Badriev and Zadvornov consider a variational inequality problem with two monotone mappings $A$ and $B$. The authors show that an iterative sequence converges weakly to a solution of the problem under suitable conditions for $A$ and $B$. In this paper, to apply a theorem in [6], we show an iterative sequence which converges strongly to a solution of the problem under same conditions for $A$ and $B$ in [3].

Keywords: Fixed point, variational inequality problem, inverse-strongly monotone mapping.

1 Introduction

Let $V$ and $H$ be Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_H$, respectively. Let $A : V \to V$ and $B : H \to H$ be mappings and $F : V \to (-\infty, \infty]$ be proper continuous lower semicontinuous functionals. Let $\Lambda : V \to H$ be a linear continuous mapping and $\Lambda^* : H \to V$ be the adjoint mapping of $\Lambda$, i.e., $\langle \Lambda^* x, v \rangle_V = \langle x, \Lambda v \rangle_H$ for all $x \in H$ and $v \in V$. In [3], Badriev and Zadvornov consider the following variational inequality problem. Find $u \in V$ such that

$$\langle Au, v - u \rangle_V + \langle \Lambda^* B \Lambda u, v - u \rangle_V + G(\Lambda v) - G(\Lambda u) + F(v) - F(u) \geq 0 \quad (1)$$

for all $v \in V$. The authors show that an iterative sequence converges weakly to a solution of the problem under suitable conditions for $A$ and $B$.

In this paper, to apply a theorem in [6], we show an iterative sequence which converges strongly to a solution of the problem under same conditions for $A$ and $B$ in [3]. The theorem in [6] is related to the result of our previous paper [1].

2 Preliminaries

Let $A : V \to V$ and $B : H \to H$ be mappings. Mappings $A : H \to H$ and $B : V \to V$ are inverse-strongly monotone mappings if there exists $\sigma_A, \sigma_B > 0$ such that

$$\langle Au - Av, u - v \rangle_V \geq \sigma_A \|Au - Av\|_V^2$$

for all $u, v \in V$ and

$$\langle Bx - By, x - y \rangle_H \geq \sigma_B \|Bx - By\|_H$$

for all $x, y \in H$. Then $A$ is called $\sigma_A$-inverse-strongly monotone and $B$ is called $\sigma_B$-inverse-strongly monotone. Let $C$ be a subset of $H$. A mapping $T$ of $C$ into itself is called nonexpansive if

$$\|Tx - Ty\|_H \leq \|x - y\|_H$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. Let $f$ be a functional
on $H$. By $P_f$, we denote the proximal mapping which takes each $x \in H$ to the element $y = P_f(x)$ that is a solution of

$$\langle y - x, z - y \rangle_H + f(z) - f(y) \geq 0$$

for all $z \in H$. $P_f$ satisfies the following.

$$\|P_f x - P_f y\|_H \leq \langle P_f x - P_f y, x - y \rangle_H$$  \hspace{1cm} (2)

for all $x, y \in H$.

To introduce our main result, we need the following theorems. Theorem 1 is the result of Badriev and Zadzorov in [3]. For the sake of completeness, we show the proof in Section 4.

**Theorem 1.** Let $V$ and $H$ be Hilbert spaces. Let $A : V \to V$ be a $\sigma_A$-inverse-stormingly monotone mapping and $B : H \to H$ be a $\sigma_B$-inverse-stormingly monotone mapping. Let $F : V \to (-\infty, \infty]$ and $G : H \to (-\infty, \infty]$ be proper convex lower semicontinuous functionals. Let $\Lambda : V \to H$ be a linear continuous mapping and $\Lambda^* : H \to V$ be the adjoint mapping of $\Lambda$, i.e., $\langle \Lambda^* x, v \rangle_V = \langle x, \Lambda v \rangle_H$ for all $x \in H$ and $v \in V$. In addition, we assume that the operator $\Lambda^*\Lambda$ is a canonical isomorphism, i.e., $v = \Lambda^*\Lambda v$ for all $v \in V$. Let $Q = V \times H \times H$ be the Hilbert space with inner product

$$\langle \cdot, \cdot \rangle_Q = \frac{1 - \tau_A}{\tau_A} \langle \cdot, \cdot \rangle_V + \frac{1}{\tau_B} \langle \cdot, \cdot \rangle_H + \frac{1}{r} \langle \cdot, \cdot \rangle_H,$$

where $\tau_A$, $\tau_B$ and $r$ are positive constants satisfying $\tau_A r < 1$. Let $T : Q \to Q$ be a mapping defined by $T_q = (T_1 q, T_2 q, T_3 q)$, where

$$T_1 q = P_{\Lambda^*\Lambda}(q_1 - \tau_A(A q_1 + \Lambda^* g_3 + r\Lambda^*(\Lambda q_1 - q_2))),$$

$$T_2 q = P_{\Lambda^*\Lambda}(q_2 - \tau_B(B q_2 - q_3 + r(q_2 - \Lambda T_1 q))),$$

$$T_3 q = q_3 + r(\Lambda T_1 q - T_2 q)$$

for $q = (q_1, q_2, q_3) \in Q$. Let $q = (u, y, \lambda)$. Then $q$ is a fixed point of $T$ if and only if

$$\begin{cases}
-Au - \Lambda^* \lambda \in \partial F(u), \\
\lambda - By \in \partial G(y), \\
y = \Lambda u.
\end{cases}$$

Moreover, $u$ is a solution of the problem (1).

Theorem 2 is the result of Badriev and Zadzorov in [3]. For the sake of completeness, we show the proof in Section 4.

**Theorem 2.** Let $V$ and $H$ be Hilbert spaces. Let $A : V \to V$ be a $\sigma_A$-inverse-stormingly monotone mapping and $B : H \to H$ be a $\sigma_B$-inverse-stormingly monotone mapping. Let $F : V \to (-\infty, \infty]$ and $G : H \to (-\infty, \infty]$ be proper convex lower semicontinuous functionals. Let $\Lambda : V \to H$ be a linear continuous mapping and $\Lambda^* : H \to V$ be the adjoint mapping of $\Lambda$, i.e., $\langle \Lambda^* x, v \rangle_V = \langle x, \Lambda v \rangle_H$ for all $x \in H$ and $v \in V$. In addition, we assume that the operator $\Lambda^*\Lambda$ is a canonical isomorphism, i.e., $v = \Lambda^*\Lambda v$ for all $v \in V$. Let $Q = V \times H \times H$ be the Hilbert space with inner product

$$\langle \cdot, \cdot \rangle_Q = \frac{1 - \tau_A}{\tau_A} \langle \cdot, \cdot \rangle_V + \frac{1}{\tau_B} \langle \cdot, \cdot \rangle_H + \frac{1}{r} \langle \cdot, \cdot \rangle_H,$$

where $\tau_A$, $\tau_B$ and $r$ are positive constants satisfying $\tau_A r < 1$,

$$\tau_A < \frac{2\sigma_A}{2\sigma_A + 1} \quad \text{and} \quad \tau_B < \frac{2\sigma_B}{2\sigma_B + 1}.$$  \hspace{1cm} (3)

Let $T : Q \to Q$ be a mapping defined by $T_q = (T_1 q, T_2 q, T_3 q)$, where $T_1 q = P_{\Lambda^*\Lambda}(q_1 - \tau_A(A q_1 + \Lambda^* g_3 + r\Lambda^*(\Lambda q_1 - q_2))),$ $T_2 q = P_{\Lambda^*\Lambda}(q_2 - \tau_B(B q_2 - q_3 + r(q_2 - \Lambda T_1 q))),$ $T_3 q = q_3 + r(\Lambda T_1 q - T_2 q)$ for $q = (q_1, q_2, q_3) \in Q$. Then $T$ is nonexpansive, i.e.,

$$\|T q - T p\|_Q \leq \|q - p\|_Q$$

for all $q, p \in Q$.

Theorem 3 is the result of Wittmann [6]. This theorem is related to results of [1].

**Theorem 3.** Let $H$ be a Hilbert space and $C$ be a closed convex subset of $H$. Let $T$ be a nonexpansive mapping of $C$ into itself such that the fixed point of $T$ is nonempty. Let $\{\alpha(k)\}$ be a sequence of $[0, 1]$ such that

$$\lim_{k \to 0} \alpha(k) = 0 \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha(k) = \infty$$

and

$$\sum_{k=0}^{\infty} |\alpha(k+1) - \alpha(k)| < \infty.$$
Let \( \{x^{(k)}\} \) be an iterative sequence of \( C \) defined as follows: \( x^{(0)} = x \in C \) and
\[
x^{(k+1)} = \alpha^{(k)} x + (1 - \alpha^{(k)}) T x^{(k)}
\]
for \( k = 0, 1, 2, \ldots \). Then \( \{x^{(k)}\} \) converges strongly to \( P_F(T)x \), where \( P_F(T)x \) is the metric projection of \( H \) onto \( F(T) \).

3 Main result

Theorem 4. Let \( V \) and \( H \) be Hilbert spaces. Let \( A : V \to V \) be a \( \sigma_A \)-inverse-stongly monotone mapping and \( B : H \to H \) be a \( \sigma_B \)-inverse-stongly monotone mapping. Let \( F : V \to (-\infty, \infty] \) and \( G : H \to (-\infty, \infty] \) be proper convex lower semicontinuous functionals. Let \( \Lambda : V \to H \) be a linear continuous mapping and \( \Lambda^* : H \to V \) be the adjoint mapping of \( \Lambda \), i.e., \( \langle \Lambda^* x, v \rangle_V = \langle x, \Lambda v \rangle_H \) for all \( x \in H \) and \( v \in V \). In addition, we assume that the operator \( \Lambda^* \Lambda \) is a canonical isomorphism, i.e., \( v = \Lambda^* \Lambda v \) for all \( v \in V \). Let \( Q = V \times H \times H \) be the Hilbert space with inner product
\[
\langle \cdot, \cdot \rangle_Q = \frac{1 - \tau_A^r}{\tau_A} \langle \cdot, \cdot \rangle_V + \frac{1}{\tau_B} \langle \cdot, \cdot \rangle_H + \frac{1}{r} \langle \cdot, \cdot \rangle_H,
\]
where \( \tau_A, \tau_B \) and \( r \) are positive constants satisfying \( \tau_A r < 1 \),
\[
\tau_A < \frac{2\sigma_A}{2\sigma_A r + 1} \quad \text{and} \quad \tau_B < \frac{2\sigma_B}{2\sigma_B r + 1}.
\]
Let \( T : Q \to Q \) be a mapping defined by \( T_q = (T_1 q, T_2 q, T_3 q) \), where
\[
T_1 q = P_{\tau_A F}(q_1 - \tau_A (A q_1 + \Lambda^* q_3 + r \Lambda^* (\Lambda q_1 - q_2))),
T_2 q = P_{\tau_B G}(q_2 - \tau_B (B q_2 + q_3 + r(q_2 - \Lambda T_1 q))),
T_3 q = q_3 + r(\Lambda T_1 q - T_2 q)
\]
for \( q = (q_1, q_2, q_3) \in Q \). Assume that \( F(T) \) is nonempty. Let \( \{q^{(k)}\} \) be the sequence constructed by \( q^{(0)} = q_0 \in Q \) and
\[
q^{(k+1)} = \alpha^{(k)} q_0 + (1 - \alpha^{(k)}) T q^{(k)}
\]
for \( k = 0, 1, 2, \ldots \), where \( \{\alpha^{(k)}\} \) be a sequence in \([0,1]\) such that
\[
\lim_{k \to 0} \alpha^{(k)} = 0 \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha^{(k)} = \infty
\]
and
\[
\sum_{k=0}^{\infty} (\alpha^{(k+1)} - \alpha^{(k)}) < \infty.
\]
Then this iterative sequence \( \{q^{(k)}\} \) converges strongly to \( q^* \) in \( Q \) as \( k \to \infty \), \( q^* \) is a fixed point of \( T \). Moreover the first component \( u \) in \( q^* = (u, y, \lambda) \) is a solution of the problem (1).

Proof. By Theorem 2, \( T \) is nonexpansive. By Theorem 3, we obtain that the sequence \( \{q^{(k)}\} \) converges strongly to a fixed point \( q^* = (u, y, \lambda) \) of \( T \). By Theorem 1, we have \( u \) is a solution of the problem \( \langle Au, v - u \rangle_V + \langle \Lambda^* BAu, v - u \rangle_V + G(Au) - G(Au) + F(v) - F(u) \geq 0 \) for all \( v \) in \( V \). This completes the proof. \( \square \)

4 Appendix

In this section, to sake of completeness, we show the proof of Theorems 1 and 2.

Proof of Theorem 1. Let \( q = (u, y, \lambda) \) be a fixed point of \( T \). Then
\[
u = T_1 q, \quad y = T_2 q, \quad \lambda = T_3 q.
\]
By the definition of \( T_3 \), we have
\[
\lambda = \lambda + r(\Lambda u - y).
\]
Then \( \Lambda u - y = 0 \), and
\[
y = \Lambda u.
\]
By the definition of \( T_1 \), we have
\[
u = P_{\tau_A F}(u - \tau_A (A u + \Lambda^* \lambda + r \Lambda^* (\Lambda u - y))).
\]
Then we have
\[
\langle u - (u - \tau_A (A u + \Lambda^* \lambda + r \Lambda^* (\Lambda u - y))), v - u \rangle_V + \tau_A F(v) - \tau_A F(u) \geq 0
\]
for all \( v \) in \( V \). Since \( \Lambda u - y = 0 \), we have
\[
\langle Au + \Lambda^* \lambda, v - u \rangle_V + F(v) - F(u) \geq 0
\]
for all \( v \) in \( V \). Hence we have
\[
F(v) \geq \langle -Au + \Lambda^* \lambda, v - u \rangle_V + F(u)
\]
for all \( v \in V \). This implies

\[
-Au - \Lambda^* \lambda \in \partial F(u).
\]

By the definition of \( T_2 \), we have

\[
y = P_{\tau_B G}(y - \tau_B(By - \lambda + r(y - \Lambda u))).
\]

Then we have

\[
\langle y - (y - \tau_B(By - \lambda + r(y - \Lambda u))), z - y \rangle_H + \tau_B G(z) - \tau_B G(y) \geq 0
\]

for all \( z \in H \). Since \( y - \Lambda u = 0 \), we have

\[
\langle By - \lambda, z - y \rangle_H + G(z) - G(y) \geq 0
\]

for all \( z \in H \). Hence we have

\[
G(z) \geq \langle By - \lambda, z - y \rangle_H + G(y)
\]

for all \( z \in H \). This implies

\[
\lambda - By \in \partial G(y).
\]

We also obtain that if \(-Au - \Lambda^* \lambda \in \partial F(u), \lambda - By \in \partial G(y) \) is a fixed point of \( T \). Let \( q = (u, y, v) \) be a fixed point of \( T \). Then we obtain that for all \( v \in V \),

\[
\langle Au + \Lambda^* \lambda, v - u \rangle_V + F(v) - F(u) \geq 0
\]

and

\[
\langle By - \lambda, \Lambda v - y \rangle_H + G(\Lambda v) - G(y) \geq 0.
\]

Since \( y = \Lambda u \), we have

\[
\langle By - \lambda, \Lambda v - y \rangle_H + G(\Lambda v) - G(y) = \langle B\Lambda u - \lambda, \Lambda v - \Lambda u \rangle_H + G(\Lambda v) - G(\Lambda u) = \langle \Lambda B\Lambda u - \Lambda^* \lambda, v - u \rangle_V + G(\Lambda v) - G(\Lambda u).
\]

Then we have

\[
\langle \Lambda^* B\Lambda u - \Lambda^* \lambda, v - u \rangle_V + G(\Lambda v) - G(\Lambda u) \geq 0
\]

for all \( v \in V \). Adding (3) and (4), we find that \( u \) satisfies the inequality (1).

To prove Theorem 2, we need the following: For all \( \epsilon > 0 \) and \( a, b \in V \), we have

\[
\langle a, b \rangle_V = \frac{1}{2\epsilon} \|a\|_V^2 + \frac{\epsilon}{2} \|b\|_V^2 - \frac{1}{2\epsilon} \|a - b\|_V^2.
\]

**Proof of Theorem 2.** Define a mapping \( S_A \) of \( V \) into \( V \) by \( S_A v = (1 - \tau_A r)v \) for \( v \in V \). Then we obtain that for all \( q_1, p_1 \in V \),

\[
\|S_A q_1 - S_A p_1\|_V^2
\]

\[
= \|(1 - \tau_A r)(q_1 - p_1) - \tau_A(Aq_1 - Ap_1)\|_V^2
\]

\[
= (1 - \tau_A r)^2 \|q_1 - p_1\|_V^2
\]

\[
- 2\tau_A (1 - \tau_A r)(Aq_1 - Ap_1, q_1 - p_1)_V
\]

\[
+ \tau_A^2 \|Aq_1 - Ap_1\|_V^2
\]

\[
\leq (1 - \tau_A r)^2 \|q_1 - p_1\|_V^2
\]

\[
- \delta_A \langle Aq_1 - Ap_1, q_1 - p_1 \rangle_V,
\]

where \( \delta_A \) is \( \tau_A (1 - \tau_A r) \left( 2 - \frac{\tau_A}{2\sigma_A (1 - \tau_A r)} \right) \). By (2) and (5) with \( \epsilon = 1 - \tau_A r, a = S_A q_1 - S_A p_1 \) and \( b = T_1 q - T_1 p \), we have

\[
\|T_1 q - T_1 p\|_V^2
\]

\[
= \|P_{\tau_A F}(S_A q_1 - \tau_A \Lambda^* (q_3 - rq_2))
\]

\[
- \tau_A F(S_A p_1 - \tau_A \Lambda^* (p_3 - rp_2))\|_V^2
\]

\[
\leq \langle T_1 q - T_1 p, S_A q_1 - S_A p_1 \rangle_V
\]

\[
- \tau_A \langle T_1 q - T_1 p, \Lambda^* (q_3 - p_3) - r \Lambda^* (q_2 - p_2) \rangle_V
\]

\[
= \frac{1}{2(1 - \tau_A r)} \|S_A q_1 - S_A p_1\|_V^2
\]

\[
+ \frac{1 - \tau_A r}{2} \|T_1 q - T_1 p\|_V^2
\]

\[
- \frac{1 - \tau_A r}{2} \|(S_A q_1 - S_A p_1) - \epsilon(T_1 q - T_1 p)\|_V^2
\]

\[
- \tau_A \langle T_1 q - T_1 p, \Lambda^* (q_3 - p_3) - r \Lambda^* (q_2 - p_2) \rangle_V
\]

for all \( q = (q_1, q_2, q_3), p = (p_1, p_2, p_3) \in Q \). Then, by (8) and (5) with \( \epsilon = 1 \), we have

\[
\frac{1}{(1 - \tau_A r)^2} \|(S_A q_1 - S_A p_1) - \epsilon(T_1 q - T_1 p)\|_V^2
\]

\[
+ \frac{1 + \tau_A r}{2\tau_A} \|T_1 q - T_1 p\|_V^2
\]

\[
\leq \frac{1}{2(1 - \tau_A r)} \|S_A q_1 - S_A p_1\|_V^2
\]

\[
- \langle T_1 q - T_1 p, \Lambda^* (q_3 - p_3) - r \Lambda^* (q_2 - p_2) \rangle_V
\]
\[
\begin{align*}
\frac{1}{2(1 - \tau_B \tau_T)} & \left\| (S_{B,q_1} - S_{A,q_1}) \right\|_V^2 \\
- & (1 - \tau_B \tau_T)\langle T_1 q - T_1 p \rangle_1^2_V \\
+ & \frac{1}{2\tau_A} \left\| T_1 q - T_1 p \right\|^2_V \\
+ & \frac{r}{2} \left\| \Lambda(T_1 q - T_1 p) \right\|_H^2 + \frac{r}{2} \left\| q_2 - p_2 \right\|^2_H \\
- & \delta_B \left\| (q_2 - p_2) \right\|^2_H
\end{align*}
\]

for all \( q, p \in Q \). Therefore we obtain that

\[
\begin{align*}
\frac{1}{2(1 - \tau_B \tau_T)} & \left\| (S_{B,q_1} - S_{A,q_1}) \right\|_V^2 \\
- & (1 - \tau_B \tau_T)\langle T_1 q - T_1 p \rangle_1^2_V \\
+ & \frac{1}{2\tau_A} \left\| T_1 q - T_1 p \right\|^2_V \\
+ & \frac{r}{2} \left\| \Lambda(T_1 q - T_1 p) \right\|_H^2 + \frac{r}{2} \left\| q_2 - p_2 \right\|^2_H \\
- & \delta_B \left\| (q_2 - p_2) \right\|^2_H
\end{align*}
\]

for all \( q, p \in Q \). Define a mapping \( S_B \) of \( H \) into \( H \) by \( S_B x = (1 - \tau_B x) x - \tau_B B x \) for \( x \in H \). Then we obtain that for all \( q_2, p_2 \in H \),

\[
\begin{align*}
\left\| S_{B,q_2} - S_{B,p_2} \right\|^2_H \\
= & \left\| (1 - \tau_B \tau_T)^2 \right\| q_2 - p_2 \right\|^2_H \\
- & 2\tau_B (1 - \tau_B \tau_T) \left\| B q_2 - B p_2, q_2 - p_2 \right\|_H \\
+ & \frac{\tau_B}{2} \left\| B q_2 - B p_2 \right\|^2_H \\
\leq & (1 - \tau_B \tau_T)^2 \left\| q_2 - p_2 \right\|^2_H \\
- & \tau_B (1 - \tau_B \tau_T) \delta_B \left\| B q_2 - B p_2, q_2 - p_2 \right\|_H,
\end{align*}
\]

where \( \delta_B = 2 - \frac{\tau_B}{\tau_B} \). By (2) and (5) with \( \epsilon = 1 - \tau_B \), \( A = S_{B,q_2} - S_{B,p_2} \) and \( b = T_2 q - T_2 p \), we have

\[
\begin{align*}
\left\| T_2 q - T_2 p \right\|^2_H \\
= & \left\| P_{\tau_B}(S_{B,q_2} + \tau_B r_1 q + \tau_B q_3) \right\|_H^2 \\
- & P_{\tau_B}(S_{B,p_2} + \tau_B r_1 p + \tau_B p_3) \right\|_H^2 \\
= & (T_2 q - T_2 p, S_{B,q_2} - S_{B,p_2})_H \\
+ & \tau_B \langle T_2 q - T_2 p, r \Lambda(T_1 q - T_1 p) \rangle + (q_3 - p_3)_H \\
\leq & \frac{1}{2(1 - \tau_B \tau_T)} \left\| S_{B,q_2} - S_{B,p_2} \right\|_H^2 \\
+ & \frac{1 - \tau_B \tau_T}{2} \left\| T_2 q - T_2 p \right\|^2_H \\
- & \frac{1}{2(1 - \tau_B \tau_T)} \left\| (S_{B,q_2} - S_{B,p_2}) \right\|_H^2 \\
- & (1 - \tau_B \tau_T) \left\| (T_2 q - T_2 p) \right\|^2_H \\
+ & \tau_B \langle T_2 q - T_2 p, r \Lambda(T_1 q - T_1 p) \rangle + (q_3 - p_3)_H
\end{align*}
\]

Then, by (8) and (5) with \( \epsilon = 1 \), we have

\[
\begin{align*}
\frac{1}{2(1 - \tau_B \tau_T)} & \left\| (S_{B,q_2} - S_{B,p_2}) \right\|_H^2 \\
- & (1 - \tau_B \tau_T)\langle T_2 q - T_2 p \rangle_1^2_H \\
+ & \frac{1 + \tau_B \tau_T}{2\tau_B} \left\| T_2 q - T_2 p \right\|^2_H \\
\leq & \frac{1}{2(1 - \tau_B \tau_T)} \left\| (S_{B,q_2} - S_{B,p_2}) \right\|_H^2 \\
+ & r \langle T_2 q - T_2 p, \Lambda(T_1 q - T_1 p) \rangle_H \\
+ & \langle T_2 q - T_2 p, q_3 - p_3 \rangle_H \\
\leq & \frac{1 - \tau_B \tau_T}{2\tau_B} \left\| q_2 - p_2 \right\|^2_H \\
- & \frac{\delta_B}{2} \left\| B q_2 - B p_2, q_2 - p_2 \right\|_H \\
+ & r \langle T_2 q - T_2 p, \Lambda(T_1 q - T_1 p) \rangle_H \\
+ & \langle T_2 q - T_2 p, q_3 - p_3 \rangle_H
\end{align*}
\]
for all $q, p \in Q$. Therefore we obtain that

\[
\frac{1}{2(1 - \tau_{B_B})\tau_B} \left\| \left( 1 - \tau_{B_B}^r \right) (q_2 - p_2) \\
- (T_2q - T_2p) - \tau_B(Bq_2 - Bp_2) \right\|^2_H
+ \frac{1 + \tau_{B_B}^r}{2\tau_B} \left\| T_2q - T_2p \right\|^2_H
+ \frac{r}{2} \left\| (T_2q - T_2p) - \Lambda(\Lambda q - \Lambda p) \right\|^2_H
+ \frac{\delta_B}{2} \left\langle Bq_2 - Bp_2, q_2 - p_2 \right\rangle_H
\leq \frac{1 - \tau_{B_B}^r}{2\tau_B} \left\| q_2 - p_2 \right\|^2_H
+ \frac{r}{2} \left\| T_2q - T_2p \right\|^2_H
+ \frac{\tau_{B_B}^r}{2} \left\| \Lambda(T_1q - T_1p) \right\|^2_H
\]

for all $q, p \in V$. For all $q, p \in V$, we have

\[
\left\| T_3q - T_3p \right\|^2_H
= \left\| q_3 - p_3 \right\|^2_H
+ 2r\left\langle q_3 - p_3, \Lambda(T_1q - T_1p) - (T_2q - T_2p) \right\rangle_H
+ r^2 \left\| \Lambda(T_1q - T_1p) - (T_2q - T_2p) \right\|^2_H.
\]

Therefore we have

\[
\frac{1}{2r} \left\| T_3q - T_3p \right\|^2_H
= \frac{1}{2r} \left\| q_3 - p_3 \right\|^2_H
+ \frac{r}{2} \left\| \Lambda(T_1q - T_1p) - (T_2q - T_2p) \right\|^2_H
\]

for all $q, p \in Q$. By (7), (9) and (10), we have

\[
\left\| Tq - Tp \right\|^2_Q + \delta_A(Aq_1 - Ap_1, q_1 - p_1)_V
+ \frac{1}{(1 - \tau_{A})\tau_A} \times
\left\| \left( 1 - \tau_{A}^r \right)(q_1 - p_1) - \tau_A(Aq_1 - Ap_1) \right\|^2_V
+ \frac{1}{(1 - \tau_{B})\tau_B} \times
\left\| \left( 1 - \tau_{B}^r \right)(q_2 - p_2) -(T_2q - T_2p) \right\|^2_H
\leq \left\| q - p \right\|^2_Q
\]

for all $q, p \in Q$. Therefore $T$ is nonexpansive.

\begin{lemma}

\end{lemma}

5 Further topic

In [2], Badrev and Zadrovorn consider a variational inequality problem for only one monotone mapping $A$. The authors show that for a strongly monotone Lipschitz continuous mapping $A$, an iterative sequence converges strongly to a solution. But using Theorem 3, we may obtain the strong convergence for an inverse-strongly monotone mapping $A$. This is a further topic.

参考文献


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